LANDMARK UNIVERSITY, OMU-ARAN

LECTURE NOTE 3

COLLEGE: COLLEGE OF SCIENCE AND ENGINEERING

DEPARTMENT: MECHANICAL ENGINEERING

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2D Convection – Complex Domains

Introduction

In practical applications of CFD, one often encounters *complex* domains. A domain is called complex when it cannot be elegantly described (or mapped) by a Cartesian grid. By way of illustration, we consider a few examples. Figure 1 shows the smallest symmetry sector of a nuclear rod bundle placed inside a circular channel of radius R. There are nineteen rods: one rod at the channel center, six rods (equally spaced) in the inner rod ring of radius b_1, and twelve rods in the outer ring of radius b_2. The rods are circumferentially equispaced. The radius of each rod is r_o. The fluid (coolant) flow is in the x_3 direction. The flow convects away the heat generated by the rods and the channel wall is insulated. It is obvious that a Cartesian grid will not fit the domain of interest because the lines of constant x_1 or x_2 will intersect the domain boundaries in an arbitrary manner. In such circumstances, it proves advantageous to adopt alternative means for mapping a complex domain.

These alternatives are to use

1. curvilinear grids or
2. finite-element-like unstructured grids.

Curvilinear Grids

It is possible to map a complex domain by means of curvilinear grids (ξ_1, ξ_2) in which directions of ξ_1 and ξ_2 may change from point to point. Also, curvilinear lines of constant ξ_1 and constant ξ_2 need not intersect orthogonally either within the domain or at the boundaries. Figure 2 shows the nineteen-rod domain of Figure 1 mapped by curvilinear grids. The figure shows that curvilinear lines generate clearly identifiable quadrilateral control volumes. When the IOCV method is used, the task is to integrate the transport equations over a typical control volume. To facilitate this, it becomes necessary to first transform the transport equations written in Cartesian coordinates to curvilinear coordinates via transformation relations:

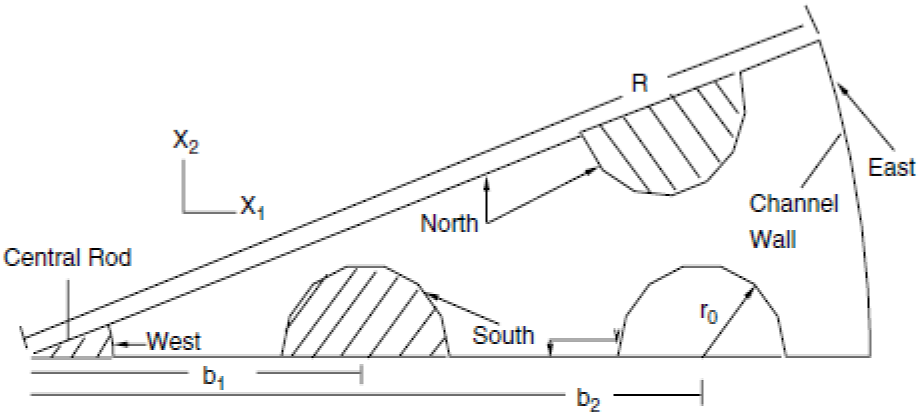

Figure 1. Example of a complex domain.

$$x_1 = F_1\,(\xi_1, \xi_2),\; x_2 = F_2\,(\xi_1, \xi_2). \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1.$$

In general, these functional relationships must be developed by *numerical* grid generation techniques (in the next lecture). The grids shown in Figure 2 are in fact generated by numerical means. For simpler domains, however, the functional relationships can be specified by algebraic functions. The new-set of transport equations in curvilinear coordinates are developed in section curvilinear Grids. One advantage of mapping domains by curvilinear grids is that one can still retain the familiar (I, J) structure to identify a node (or the corresponding control volume) because, as can be seen from Figure 2, along any curvilinear line ξ_1, the total number of intersections with constant-ξ_2 lines remains constant and vice versa. Further advantages of this identifying structure will become clear in section curvilinear Grids.

Unstructured Grids

Another alternative for a complex domain is to map the domain by triangles or any *n*-sided polygons (including quadrilaterals) or any mix of triangles and polygons. Figure 3 shows the mapping of a nineteen-rod bundle by triangles as an example. In this case, the rods are arranged in such a way that the smallest symmetry sector is a *doubly connected* domain.

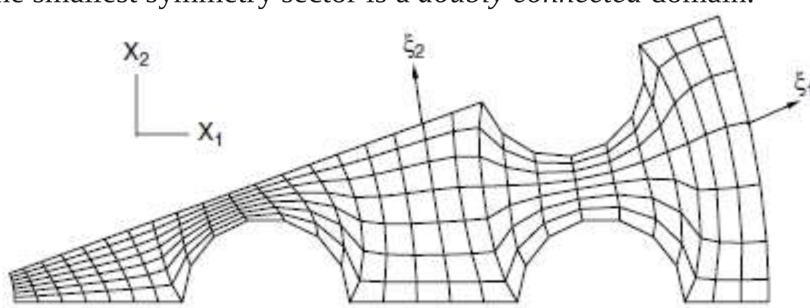

Figure 2. Nineteen-rod bundle – curvilinear grids.

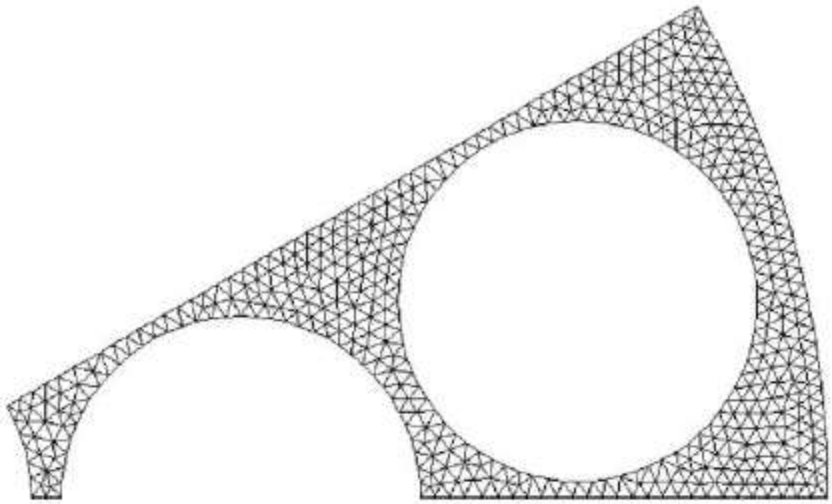

Figure 3. Nineteen-rod bundle – unstructured grid.

Such mapping can be generated by commercially available grid generators such as ANSYS. Each triangle may now be viewed as a *control volume* over which the transport equations are to be integrated to arrive at the discretised equations. It will be recognized that a triangle is a very convenient elemental construct because it can map any convex intrusion or concave extrusion at the domain boundaries. More importantly, triangles can also effectively skirt any *blocked* region within the overall domain, as shown in Figure 3. Such skirting cannot be elegantly accomplished if curvilinear grids are used for mapping. The flexibility offered by mapping by triangulation is thus obvious. Further, it is not necessary that all triangles be of the same size or shape. In spite of this flexibility, it becomes necessary to make a significant departure from curvilinear grid practice with respect to *node identification* when unstructured grids are used. It is obvious from Figure 3, for example, that one cannot readily identify elements (or nodes) by employing the familiar (I, J) structure as was possible with curvilinear grids. Elements, perforce, must be identified serially with a single identifier N (say). Thus, an element having identifier N will interact with elements having arbitrary identifying numbers *without* any generalisable rules. This contrasts with the case of curvilinear grids in which a control volume (I, J) will always interact with control volumes identified by $(I + 1, J)$, $(I - 1, J)$, $(I, J + 1)$, and $(I, J - 1)$. This serial numbering has consequences for solution of discretised equations evolved on an unstructured grid. In passing, we note that there are a variety of methods for triangulation. Automatic triangulation requires detailed considerations from the subject of computational geometry. In next lecture, some simpler approaches will be introduced. Most CFD practitioners, however, employ commercially available packages such as ANSYS for unstructured grid generation.

Curvilinear Grids

Coordinate Transformation

Our first task is to transform the transport equations in Cartesian coordinates to those in curvilinear coordinates. Thus, employing the chain rule, we can write the first-order derivatives as

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi_1}{\partial x_1}\frac{\partial}{\partial \xi_1} + \frac{\partial \xi_2}{\partial x_1}\frac{\partial}{\partial \xi_2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad 2$$

$$\frac{\partial}{\partial x_2} = \frac{\partial \xi_1}{\partial x_2}\frac{\partial}{\partial \xi_1} + \frac{\partial \xi_2}{\partial x_2}\frac{\partial}{\partial \xi_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad 3$$

The next task is to determine derivatives of ξ_1 and ξ_2 with respect to x_1 and x_2 knowing equations 1. To do this, we note that

$$dx_1 = \frac{\partial x_1}{\partial \xi_1} d\xi_1 + \frac{\partial x_1}{\partial \xi_2} d\xi_2, \quad \ldots\ldots\ldots\ldots\ldots.. \quad 4$$
$$dx_2 = \frac{\partial x_2}{\partial \xi_1} d\xi_1 + \frac{\partial x_2}{\partial \xi_2} d\xi_2, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 5$$

These relations can be written in matrix form as $|dx| = |A||d\xi|$, or

$$\begin{vmatrix} dx_1 \\ dx_2 \end{vmatrix} = \begin{vmatrix} \partial x_1/\partial \xi_1 & \partial x_1/\partial \xi_2 \\ \partial x_2/\partial \xi_1 & \partial x_2/\partial \xi_2 \end{vmatrix} \begin{vmatrix} d\xi_1 \\ d\xi_2 \end{vmatrix}. \quad \ldots\ldots\ldots\ldots\ldots \quad 6$$

Now, manipulation of Equations 4 and 5 will show that

$$d\xi_1 = \frac{1}{Det\ A}\left[cof\left(\frac{\partial x_1}{\partial \xi_1}\right) dx_1 + cof\left(\frac{\partial x_2}{\partial \xi_1}\right) dx_2\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 7$$
$$d\xi_2 = \frac{1}{Det\ A}\left[cof\left(\frac{\partial x_1}{\partial \xi_2}\right) dx_1 + cof\left(\frac{\partial x_2}{\partial \xi_2}\right) dx_2\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 8$$

Where cof denotes *cofactor of* and Det A stands for *determinant of* A. Thus, from the last two equations, it is easy to deduce that

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1}{Det\ A} cof\left(\frac{\partial x_1}{\partial \xi_1}\right) = \frac{1}{Det\ A}\left(\frac{\partial x_2}{\partial \xi_1}\right) = \frac{\beta_1^1}{Det\ A} \quad \ldots\ldots\ldots\ldots \quad 9$$

$$\frac{\partial \xi_1}{\partial x_2} = \frac{1}{Det\ A} cof\left(\frac{\partial x_2}{\partial \xi_1}\right) = -\frac{1}{Det\ A}\left(\frac{\partial x_1}{\partial \xi_2}\right) = \frac{\beta_1^2}{Det\ A} \quad \ldots\ldots\ldots\ldots\ldots \quad 10$$

$$\frac{\partial \xi_2}{\partial x_1} = \frac{1}{Det\ A} cof\left(\frac{\partial x_1}{\partial \xi_2}\right) = -\frac{1}{Det\ A}\left(\frac{\partial x_2}{\partial \xi_1}\right) = \frac{\beta_2^1}{Det\ A} \quad \ldots\ldots\ldots\ldots\ldots \quad 11$$
$$\frac{\partial \xi_2}{\partial x_2} = \frac{1}{Det\ A} cof\left(\frac{\partial x_2}{\partial \xi_2}\right) = \frac{1}{Det\ A}\left(\frac{\partial x_1}{\partial \xi_1}\right) = \frac{\beta_2^2}{Det\ A} \quad \ldots\ldots\ldots\ldots\ldots.. \quad 12$$

Where the βs are called the *geometric coefficients* and are given by

$$\beta_1^1 = \frac{\partial x_2}{\partial \xi_2}, \ \ \beta_1^2 = -\frac{\partial x_1}{\partial \xi_2}, \ \ \beta_2^1 = -\frac{\partial x_2}{\partial \xi_1}, \ \ \beta_2^2 = \frac{\partial x_1}{\partial \xi_1} \quad \ldots\ldots\ldots\ldots \quad 13$$

Further, it follows that

$$Det\ A = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} = \beta_1^1\,\beta_2^2 - \beta_2^1\beta_1^2 = J, \quad \ldots\ldots\ldots\ldots.. \quad 14$$

Where symbol J stands for the *Jacobian* of the matrix A. We can now rewrite Equations 2 and 3 as

$$\frac{\partial}{\partial x_1} = \frac{1}{J}\left[\beta_1^1 \frac{\partial}{\partial \xi_1} + \beta_2^1 \frac{\partial}{\partial \xi_2}\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 15$$

$$\frac{\partial}{\partial x_2} = \frac{1}{J}\left[\beta_1^2 \frac{\partial}{\partial \xi_1} + \beta_2^2 \frac{\partial}{\partial \xi_2}\right]. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \ 16.$$